Mathematical Induction

(Mathematically reasoning from effects to causes)

An Introduction

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Contents

To parents and other teachers,

We study the topic of mathematical induction because it is a powerful tool that can be used to prove various theses. The idea of proof in mathematics is a very rigorous one. Something that can be proven is not open for debate or the whim of private opinion once the basis on which the proof is made has been established. People might argue about whether a given basis or starting point is valid or not, but once such a thing is established, that which flows logically from it can only be denied by the stupid or the insane. Don't chuckle too loudly at this, there are many people out there who deny what can be proven and still insist that they are neither stupid nor insane. Take for example the atheistic scientist who believes his discipline, along with all true disciplines, are something that can be studied using the scientific method. The scientific method (which boils down to saying that things happen for a reason) ought to form their basis or starting point for any scientific investigation. Given this basis, for a scientist to deny something proven using reason alone means that he arbitrarily abandons the scientific method. This is either stupid or insane, yet it is exactly what happens when scientists study St. Thomas Aquinas' proofs for God's existence and end up rejecting them. They become like the parson in Oliver Goldsmith's poem *The Deserted Village* of whom it is said:

"In arguing, too, the parson own'd his skill,
For, e'en though vanquished, he could argue still."

It is hoped that by having your children or other students study this method of proof they will gain a deeper understanding and respect for the power of human reason properly applied and then follow the path of reason to its fullest and deepest reality – the search for truth – and hence to God alone.

To students,

Pay close attention to this topic. It isn't as tricky as it first may appear, and it certainly isn't as scary as its fancy name implies. Although, as with most topics, rock solid algebra skills are an essential prerequisite, there's no reason why you can't do this topic at any convenient time in your Year 11 and 12 courses. This topic looks at a way of proving things that is a little more subtle than the ways you might have seen before now. By now you would have seen proofs in the area of deductive geometry (for example proving that two triangles are congruent, or proving some theorem to do with the geometry of a circle). Deductive proofs argue from an axiom (something which we accept as true) to a result. In other words, they argue from causes to effects. The only axioms in an inductive proof are the laws of algebra and an assumption of logic. The difference with this sort of proof is that it argues from what can be observed to what must (that's right – MUST!) be true even though it cannot be observed directly. In other words, it argues from effects to causes. Work hard to master this technique (as well as your deductive geometry), because the ability to think logically is under attack nowadays. With a habit of logical thinking you're less likely to get duded by a dodgy argument, and that's a good thing, because dodgy arguments produce stupid ideas like atheism, which would be a terrible thing to fall victim to.

Even though there's only about a dozen pages in this handout don't think it can be read is a short time. Work through the ideas slowly and carefully, doing your best to understand everything as you go. As usual, ask as many questions as needed to ensure this understanding.

Scaffold for Proofs Involving Series

One part of mathematical induction that students should appreciate is that just about every proof follows the exact same pattern. We can even set up a "scaffold" that all inductive proofs can be built around that looks something like this[1]:

Test the truth of the statement for $n=1$ (or whatever the first value of n is)

$$\begin{aligned} LHS &= \ldots \\ &= \vdots \\ RHS &= \ldots \\ &= \vdots \\ &= LHS \end{aligned}$$

$\therefore$ the statement is true for $n=1$ (or whatever the first value of n is)

Assume that the statement is true for some value $n=k$.[2]

That is, [here we write the statement we originally set out to prove with k written in place of n].

We now consider the truth of the statement when $n=k+1$. We aim to prove that the statement is true for $n=k+1$ given that it is true for $n=k$.

That is, we are r.t.p.[3] that [here we write the statement with $k+1$ in place of n].

Proof:[4]

$$\begin{aligned} LHS &= \text{[the left hand side of the previous line]} \\ &= \text{[the statement with } k \text{ in place of } n \text{ is substituted for its corresponding part of the previous line]} \\ &= \text{[the expression is simplified as much as possible]} \\ &= \text{[the expression is manipulated algebraically until it matches the right hand side]} \\ &= RHS \end{aligned}$$

Hence, if the statement is true for $n=k$ then it is also true for $n=k+1$. Since the statement is true for $n=1$ (or whatever the first value of n is) then it is true for $n=2$ (or whatever the second value of n is) and so on and therefore by the principle of Mathematical Induction the statement is true for all n . Q.E.D.[5]

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[1] Don't let any of the notation scare you. Your teacher will explain it all, and none of it is terribly difficult. Remember, new does not mean hard!

[2] Many teachers simply use the statement "Assume true for *n = k*". I'm not a fan of this type of wording because many people fail to understand that this method of proof is not simply committing the logical error of *begging the question* and then fail to pay due respect to this form of proof. My setting out (which you will see in the examples that follow) is a little more subtle, but avoids this terminology so that people (hopefully) see the rigor in this method. This is important, because it is the same sort of argument (and hence the same rigor) that we find in St. Thomas Aquinas' third proof for God's existence, which is handy to have mastered when confounding the foolishness of atheism or some forms of radical sola fideism.

[3] r.t.p. stands for "required to prove" and is a standard abbreviation in mathematics.

[4] LHS and RHS stand for "left hand side" and "right hand side" respectfully. They refer to the sides of the expression being considered in the problem.

[5] Q.E.D. stands for "quod erat demonstrandum" which is Latin for "[This is] that which was to be proven".

Test For Making Sure You "Get" Mathematical Induction

The whole method of mathematical induction depends on the idea that the statement is true for $n=k+1$ if it is true for $n=k$. This is not the same as saying we test for one value of n and the next value of n and see that they are both true! What we do in mathematical induction is show USING the truth of the statement for some value of n that is it is true for the next value of n. Re-read the previous sentences of this paragraph and try writing (all of) them in your own words, check with your teacher or someone rigorously logical[6] to see if your understanding is correct. For the philosophically minded, inductive proofs argue from effects to causes: we see what is true for some aspects of a thing and from those truths argue back to a necessary condition – this is precisely how St. Thomas Aquinas' proofs for God's existence work. It doesn't matter if you don't get the philosophical applications now, maths tests won't examine you on this aspect, what is important however is that you know that St. Paul wasn't wrong when in Romans 1:20 he chides pagans for failing to acknowledge God when His existence can be known from "things that are made"[7] and that you shouldn't make the same mistakes as they did! Anyhow, you can do the re-wording of those earlier underlined sentences here:

[6] Here's a simple test for rigorous logic: The person isn't an atheist and a scientist at the same time. He or she also thinks that Darwinist macro evolution is impossible not because faith tells them so, but because it is a philosophical impossibility in and of itself. They are also most likely devoted to the philosophy of St. Thomas Aquinas and Aristotle, think that phenomenology is hugely over rated and are also big fans of G. K. Chesterton.

[7] When you get the time, read this quote in its context. A careful reading of St. Paul's Letter to the Romans (chapter 1, verses 18-32) should provide a genuine eye opening experience of many of the errors of our own age for the aware, intelligent reader!

Sigma Notation

Before jumping into examples of proof by mathematical induction, a short introduction to sigma notation by way of a few examples may be needed.

$\sum$ is the uppercase Greek letter "sigma". (It's pronounced cig (as in cigarette) ma (as in march)). It is used to denote a sum of a series of numbers or terms.

$$\sum_{r=a}^{n}(f(r)) \text{ is read as "The sum, from } r=a \text{ to } n\text{, of } f(r)\text{"}$$

Example 1:

$$\sum_{r=1}^{4}\left(r^2+3r-1\right) \text{ is the sum, from } r=1 \text{ to } 4\text{, of } r^2+3r-1\text{.}$$

That is:

$$\begin{aligned}\sum_{r=1}^{4}\left(r^2+3r-1\right) &= \underbrace{\left(1^2+3\times1-1\right)}_{r=1}+\underbrace{\left(2^2+3\times2-1\right)}_{r=2}+\underbrace{\left(3^2+3\times3-1\right)}_{r=3}+\underbrace{\left(4^2+3\times4-1\right)}_{r=4}\\ &=(1+3-1)+(4+6-1)+(9+9-1)+(16+12-1)\\ &=3+9+17+27\\ &=56\end{aligned}$$

You don't need to label your substitutions $\underbrace{}_{r=1}$, $\underbrace{}_{r=2}$, etc. It's only done here to show you how the substitutions work.

Example 2:

$$\sum_{r=1}^{n}\left(r^2+r\right) \text{ is the sum, from } r=1 \text{ to } n\text{, of } r^2+r\text{.}$$

That is:

$$\sum_{r=1}^{n}\left(r^2+r\right)=\left(1^2+1\right)+\left(2^2+2\right)+\left(3^2+3\right)+\cdots+\left(n^2+n\right)$$

Try a few sigma notation exercises from your textbook before going into the next section. It's not really that hard, just new.

Mathematical Induction Examples

One example each of series (one using sigma notation, one in regular notation), divisibility and inequality follows. Keep the explanation you wrote down two pages ago in mind as you work through the examples that follow.

Mathematical Induction Proof Using Sigma Notation

Example 1: Prove that $\sum_{r=1}^{n} r = \frac{1}{2}\left(n^2 + n\right)$

Proof:

Testing the statement for $n=1$ we have:

$$\begin{aligned} LHS &= \sum_{r=1}^{1} r \\ &= 1 \end{aligned}$$

$$\begin{aligned} RHS &= \frac{1}{2}\left(1^2 + 1\right) \\ &= \frac{1}{2}(2) \\ &= 1 \\ &= LHS \end{aligned}$$

Hence the statement is true for $n=1$.

Since there is a value of n for which the statement if true, say that this value is $n=k$.

That is, $\sum_{r=1}^{k} r = \frac{1}{2}\left(k^2 + k\right)$.

That is, $1+2+\cdots+k = \frac{1}{2}\left(k^2 + k\right)$.

We now consider the case where $n=k+1$ and try to prove the truth of the statement for this value given its truth for $n=k$.

That is we aim to prove $\sum_{r=1}^{k+1} r = \frac{1}{2}\left((k+1)^2 + (k+1)\right)$

That is, we are r.t.p. that $1+2+\cdots+k+(k+1) = \frac{1}{2}\left((k+1)^2 + (k+1)\right)$

That is, we are r.t.p. that $1+2+\cdots+k+(k+1)=\frac{1}{2}\left(k^2+2k+1+k+1\right)$

That is, we are r.t.p. that $1+2+\cdots+k+(k+1)=\frac{1}{2}\left(k^2+3k+2\right)$

That is, we are r.t.p. that $1+2+\cdots+k+(k+1)=\frac{1}{2}(k+2)(k+1)$

Proof:

$$\begin{aligned} LHS &= 1+2+\cdots+k+(k+1) \\ &= \frac{1}{2}\left(k^2+k\right)+(k+1) \\ &= \frac{1}{2}k^2+\frac{3}{2}k+1 \\ &= \frac{1}{2}\left(k^2+3k+2\right) \\ &= \frac{1}{2}(k+2)(k+1) \\ &= RHS \end{aligned}$$

Hence, if the statement is true for $n=k$ then it is true for $n=k+1$. Since the statement is true for $n=1$ then it is true for $n=2$ and so on, and by the principle of Mathematical Induction the statement is true for all n. Q.E.D.

Mathematical Induction Proof Using Series

Example 2: Prove that $1+3+5+\cdots+(2n-1)=n^2$

Proof:

Testing the statement for $n=1$ we have:

$$\begin{aligned} LHS &= 2\times 1-1 \\ &= 1 \\ RHS &= 1^2 \\ &= 1 \\ &= LHS \end{aligned}$$

Hence the statement is true for $n=1$.

Since there is a value of n for which the statement if true, say that this value is $n=k$.

That is, $1+3+5+\cdots+(2k-1)=k^2$.

We now consider the case where $n=k+1$ and try to prove the truth of the statement for this value given its truth for $n=k$.

That is we aim to prove $1+3+5+\cdots+(2k-1)+(2(k+1)-1)=(k+1)^2$

That is, we are r.t.p. that $1+3+5+\cdots+(2k-1)+(2k+1)=(k+1)^2$

Proof:

$$\begin{aligned} LHS &= 1+3+5+\cdots+(2k-1)+(2k+1) \\ &= k^2+(2k+1) \\ &= k^2+2k+1 \\ &= (k+1)^2 \\ &= RHS \end{aligned}$$

Hence, if the statement is true for $n=k$ then it is true for $n=k+1$. Since the statement is true for $n=1$ then it is true for $n=2$ and so on, and by the principle of Mathematical Induction the statement is true for all n . Q.E.D.

Mathematical Induction Proof Using Divisibility

Example 3: Prove that $3^{4n}-1$ is divisible by 80 for all positive integral values of n

Proof:

Testing the statement for $n=1$ we have:

$$\begin{aligned}3^{4\times 1}-1&=3^4-1\\&=81-1\\&=80\\&=80M \text{ where } M \text{ is a whole number.}\end{aligned}$$

Hence the statement is true for $n=1$.

Since there is a value of n for which the statement if true, say that this value is $n=k$.

That is, $3^{4\times k}-1=80N$ where N is a whole number.

That is, $3^{4k}-1=80N$ where N is a whole number.

We now consider the case where $n=k+1$ and try to prove the truth of the statement for this value given its truth for $n=k$.

That is we aim to prove $3^{4(k+1)}-1=80P$ where P is a whole number.

That is, we are r.t.p. that $3^{4k+4}-1=80P$ where P is a whole number.

Proof:[8]

$$\begin{aligned}LHS&=3^{4k+4}-1\\&=3^{4k}\times 3^4-1\\&=3^4\left(3^{4k}-1\right)+3^4-1\\&=3^4\left(\left(3^{4k}-1\right)+1\right)+1\\&=3^4(80N+1)-1\\&=3^4(80N)+3^4-1\\&=3^4(80N)+80\\&=80\left(3^4N+1\right)\\&=80P \text{ since the integers are closed under addition.}\\&=RHS\end{aligned}$$

[8] The expression "integers are closed under addition" means that when we add integers or multiples of integers we always end up with answers which themselves are integers.

Hence, if the statement is true for $n=k$ then it is true for $n=k+1$. Since the statement is true for $n=1$ then it is true for $n=2$ and so on, and by the principle of Mathematical Induction the statement is true for all $n \geq 1$. Q.E.D.

Mathematical Induction Proof Using Inequalities

Example 4: Prove that $n! > 2^n$ for all $n \geq 4$

Proof:

Testing the statement for $n=4$ we have:

$$\begin{aligned} LHS &= 4! \\ &= 24 \\ RHS &= 2^4 \\ &= 16 \\ &\therefore LHS > RHS \end{aligned}$$

Hence the statement is true for $n=4$.

Since there is a value of n for which the statement if true, say that this value is $n=k$.

That is, $k! > 2^k$

We now consider the case where $n=k+1$ and try to prove the truth of the statement for this value given its truth for $n=k$.

That is we aim to prove $(k+1)! > 2^{k+1}$

Proof:

$$\begin{aligned} &\text{Begining with } k! > 2^k \\ &\text{and seeing that } (k+1) > 2 \text{ when } k > 4 \\ &\text{we are left with } k! \times (k+1) > 2^k \times 2 \\ &\text{i.e. } (k+1)! > 2^{k+1} \end{aligned}$$

Hence, if the statement is true for $n=k$ then it is true for $n=k+1$. Since the statement is true for $n=4$ then it is true for $n=5$ and so on, and by the principle of Mathematical Induction the statement is true for all $n \geq 4$. Q.E.D.

The usual rules now apply

The usual rules are:

1. Find a good textbook with lots of questions.
2. Do them.
3. Make sure you get them right.
4. Do as many examples from past examinations papers as you can get your hands on.
5. Make sure you get them right.
6. Ask as many questions as you need along the way.

More Examples (from past HSC Examination papers)

The following examples were selected to only include examples that only require sound algebra as a prerequisite.

2008 Extension I HSC Examination Q3 (b)

Use mathematical induction to prove that, for integers $n \geq 1$,

$$1\times3+2\times4+3\times5+\cdots+n(n+2)=\frac{n}{6}(n+1)(2n+7).$$

Proof:

Testing the statement for $n=1$ we have:

$$\begin{aligned} LHS &= 1(1+2) \\ &= 3 \\ RHS &= \frac{1}{6}(1+1)(2\times1+7) \\ &= \frac{1}{6}\times2\times9 \\ &= 3 \\ &= LHS \end{aligned}$$

Hence the statement is true for $n=1$.

Since there is a value of n for which the statement if true, say that this value is $n=k$.

That is, $1\times3+2\times4+3\times5+\cdots+k(k+2)=\frac{k}{6}(k+1)(2k+7)$.

We now consider the case where $n=k+1$ and try to prove the truth of the statement for this value given its truth for $n=k$.

That is we aim to prove

$$1\times3+2\times4+3\times5+\cdots+k(k+2)+(k+1)((k+1)+2)=\frac{(k+1)}{6}((k+1)+1)(2(k+1)+7)$$

That is, we are r.t.p. that $1\times3+2\times4+3\times5+\cdots+k(k+2)+(k+1)(k+3)=\frac{(k+1)}{6}(k+2)(2k+9)$

Proof:

$$\begin{aligned}
LHS &= 1\times3+2\times4+3\times5+\cdots+k(k+2)+(k+1)(k+3)\\
&= \frac{k}{6}(k+1)(2k+7)+(k+1)(k+3)\\
&= (k+1)\left[\frac{k}{6}(2k+7)+(k+3)\right]\\
&= \frac{1}{6}(k+1)[k(2k+7)+6(k+3)]\\
&= \frac{(k+1)}{6}\left(2k^2+7k+6k+18\right)\\
&= \frac{(k+1)}{6}\left(2k^2+13k+18\right)\\
&= \frac{(k+1)}{6}(k+2)(2k+9)
\end{aligned}$$

Hence, if the statement is true for $n=k$ then it is true for $n=k+1$. Since the statement is true for $n=1$ then it is true for $n=2$ and so on, and by the principle of Mathematical Induction the statement is true for all $n \geq 1$. Q.E.D.

2007 Extension I HSC Examination Q4 (b)

Use mathematical induction to prove that $7^{2n-1}+5$ is divisible by 12, for all positive integers $n \geq 1$.

Proof:

Testing the statement for $n=1$ we have:

$$
\begin{aligned}
7^{2\times1-1}+5 &= 7+5 \\
&= 12 \text{ which is divisible by } 12.
\end{aligned}
$$

Hence the statement is true for $n=1$.

Since there is a value of n for which the statement if true, say that this value is $n=k$.

That is, $7^{2k-1}+5=12N$ where N is a whole number.

We now consider the case where $n=k+1$ and try to prove the truth of the statement for this value given its truth for $n=k$.

That is we aim to prove $7^{2(k+1)-1}+5=12P$ where P is a whole number.

That is, we are r.t.p. that $7^{2k+1}+5=12P$ where P is a whole number.

Proof:

$$
\begin{aligned}
LHS &= 7^{2k+1}+5 \\
&= 7^2 \times 7^{2k-1}+5 \\
&= 49 \times 7^{2k-1}+5 \\
&= 49 \times \left(7^{2k-1}+5\right) - 48 \times 5 \\
&= 49 \times 12N - 48 \times 5 \\
&= 12\left(49N - 4 \times 5\right) \\
&= 12P \text{ where } P \text{ is a whole number.}
\end{aligned}
$$

Hence, if the statement is true for $n=k$ then it is true for $n=k+1$. Since the statement is true for $n=1$ then it is true for $n=2$ and so on, and by the principle of Mathematical Induction the statement is true for all $n \geq 1$. Q.E.D.

2005 Extension I HSC Examination Q4 (d)

Use the principle of mathematical induction to show that $4^n - 1 - 7n > 0$ for all positive integers $n \geq 2$.

Proof:

Testing the statement for $n = 2$ we have:

$$\begin{aligned} 4^2 - 1 - 7 \times 2 &= 16 - 1 - 14 \\ &= 1 \\ &> 0 \end{aligned}$$

Hence the statement is true for $n = 2$.

Since there is a value of n for which the statement if true, say that this value is $n = k$.

That is, $4^k - 1 - 7k > 0$

We now consider the case where $n = k + 1$ and try to prove the truth of the statement for this value given its truth for $n = k$.

That is we aim to prove $4^{(k+1)} - 1 - 7(k+1) > 0$

That is, we are r.t.p. that $4^{(k+1)} - 8 - 7k > 0$

Proof:

$$\begin{aligned} 4^{(k+1)} - 8 - 7k &= 4 \times 4^k - 1 - 7k - 7 \\ &= 4 \times \left(4^k - 1 - 7k\right) - 7 + 3 + 3 \times 7k \\ &= 4 \times \left(4^k - 1 - 7k\right) - 4 + 21k \\ &= 4 \times \left(4^k - 1 - 7k\right) + 21k - 4 \end{aligned}$$

Now, since $k \geq 2,\ 21k - 4 > 0$ and since $\left(4^k - 1 - 7k\right) > 0$

$$4 \times \left(4^k - 1 - 7k\right) + 21k - 4 > 0$$

that is $4^{(k+1)} - 8 - 7k > 0$

Hence, if the statement is true for $n = k$ then it is true for $n = k + 1$. Since the statement is true for $n = 2$ then it is true for $n = 3$ and so on, and by the principle of Mathematical Induction the statement is true for all $n \geq 2$. Q.E.D.

2004 Extension I HSC Examination Q4 (a)

Use mathematical induction to prove that for all integers $n \geq 3$,

$$\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\cdots\left(1-\frac{2}{n}\right)=\frac{2}{n(n-1)}.$$

Proof:

Testing the statement for $n=3$ we have:

$$\begin{aligned} LHS &= 1-\frac{2}{3} \\ &= \frac{1}{3} \\ RHS &= \frac{2}{3(3-1)} \\ &= \frac{1}{3} \\ &= LHS \end{aligned}$$

Hence the statement is true for $n=3$.

Since there is a value of n for which the statement if true, say that this value is $n=k$.

That is, $\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\cdots\left(1-\frac{2}{k}\right)=\frac{2}{k(k-1)}$.

We now consider the case where $n=k+1$ and try to prove the truth of the statement for this value given its truth for $n=k$.

That is we aim to prove $\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\cdots\left(1-\frac{2}{k}\right)\left(1-\frac{2}{(k+1)}\right)=\frac{2}{(k+1)((k+1)-1)}$

That is, we are r.t.p. that $\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\cdots\left(1-\frac{2}{k}\right)\left(1-\frac{2}{k+1}\right)=\frac{2}{k(k+1)}$

Proof:

$$
\begin{aligned}
LHS &= \left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\cdots\left(1-\frac{2}{k}\right)\left(1-\frac{2}{k+1}\right) \\
&= \frac{2}{k(k-1)}\left(1-\frac{2}{k+1}\right) \\
&= \frac{2}{k(k-1)}\left(\frac{k+1}{k+1}-\frac{2}{k+1}\right) \\
&= \frac{2}{k(k-1)}\left(\frac{k-1}{k+1}\right) \\
&= \frac{2}{k(k-1)}\times\frac{(k-1)}{(k+1)} \\
&= \frac{2}{k(k+1)}
\end{aligned}
$$

Hence, if the statement is true for $n=k$ then it is true for $n=k+1$. Since the statement is true for $n=3$ then it is true for $n=4$ and so on, and by the principle of Mathematical Induction the statement is true for all $n \geq 3$. Q.E.D.

2003 Extension I HSC Examination Q3 (d)

Use mathematical induction to prove that $\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$ for all positive integers n.

Proof:

Testing the statement for $n=1$ we have:

$$\begin{aligned} LHS &= \frac{1}{(2\times1-1)(2\times1+1)} \\ &= \frac{1}{1\times3} \\ &= \frac{1}{3} \\ RHS &= \frac{1}{2\times1+1} \\ &= \frac{1}{3} \\ &= LHS \end{aligned}$$

Hence the statement is true for $n=1$.

Since there is a value of n for which the statement if true, say that this value is $n=k$.

That is, $\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$.

We now consider the case where $n=k+1$ and try to prove the truth of the statement for this value given its truth for $n=k$.

That is we aim to prove

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}=\frac{(k+1)}{2(k+1)+1}$$

That is, we are r.t.p. that $\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$

Proof:

$$
\begin{aligned}
LHS &= \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)} \\
&= \frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} \\
&= \frac{k(2k+3)}{(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)} \\
&= \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\
&= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} \\
&= \frac{k+1}{2k+3}
\end{aligned}
$$

Hence, if the statement is true for $n=k$ then it is true for $n=k+1$. Since the statement is true for $n=1$ then it is true for $n=2$ and so on, and by the principle of Mathematical Induction the statement is true for all n. Q.E.D.